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Controlled Dense Coding with a Four-Particle Non-maximally Entangled State

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Abstract Two schemes for controlled dense coding with a four-particle entangled state are investigated, one with entanglement concentration and the other with generalized measurement. In these protocols, the supervisor (Cliff) can control the average amount of information transmitted from the sender (Alice and David) to the receiver (Bob) only by adjusting his local measurement angle θ. It is shown that the results for the average amounts of information are unique from the different two schemes.

Keywords Controlled dense coding · Four-particle entangled state · Average amount of information · Entanglement concentration · POVM

1 Introduction

Quantum entanglement not only holds the power for demonstration of the quantum nonlocality against local hidden variable theory [1], but also provides promising and wide applications in quantum information processing, such as quantum teleportation [2–5], quantum dense coding [6, 7], quantum state sharing [8–12], quantum information splitting [13–17], and so on. Dense coding, or superdense coding, first proposed by Bennett and Wiesner, is one of the important branches of quantum information theory. In the original protocol, the authors have showed how entangled states can increase the communication capacity between the sender and the receiver [6]. It is shown that if the sender and the receiver preshare a maximally bipartite entangled state, the sender can transmit 2 bits of information by manipulating and sending only one qubit [6, 7]. Quantum dense coding was experimentally implemented by Mattle et al. in an optical system [18], and then by Fang et al. by the use of NMR techniques [19].

The initial controlled dense coding protocol was proposed in 2001 by Hao and Guo [20], where one party (Alice) can transmit information to the second party (Bob) whereas the local measurement of the third party (Cliff) serves as quantum erasure. Cliff controls the quantum

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channel between Alice and Bob and the average amount of information transmitted from Alice to Bob via a local measurement. This protocol was experimentally demonstrated by Zhang and Jing for continuous variables [21, 22]. Since that, Jiang et al. proposed a scheme to realize controlled dense coding with three-particle symmetric state [23]. Huang et al. studied controlled dense coding scheme between multi-parties with multi-qubit GHZ state [24].

In this paper, two methods are shown to realize controlled dense coding with a four-particle non-maximally entangled state. The first strategy is that Alice first concentrates the entanglement of the channel between the senders and the receiver Bob, and then performs dense coding. The second one is that Alice applies directly one of the four unitary operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her particle and then sends it to Bob, while David applies one of the two unitary operators $\{I, \sigma_X\}$ on his particle and also sends it to Bob. Then Bob can obtain 3 bits of information with a certain probability via performing a projective measurement and a generalized measurement described by positive-operator-valued measure (POVM) elements on his three-particle states [25]. It is also shown that the average amounts of information transmitted from Alice and David to Bob are unique from the different two strategies.

2 Dense Coding with Entanglement Concentration

Now we suppose that the four particles 1, 2, 3 and 4, held by the senders Alice, David, the receiver Bob and the supervisor Cliff, respectively, are in a non-maximally entangled state

$$|\xi\rangle_{1234} = \alpha|0000\rangle_{1234} + \beta|0001\rangle_{1234} + \gamma|1110\rangle_{1234} + \delta|1111\rangle_{1234}. \tag{1}$$

where, without loss of generality, the coefficients α, γ, β and δ are supposed to be real and $\alpha \geq \gamma$ and $\beta \geq \delta$. In order to control the amount of information transmitted from the senders to Bob, Cliff performs a von Neumann measurement on his particle 4 under the basis

$$|+\rangle_4 = \sin\theta|0\rangle_4 + \cos\theta|1\rangle_4, \qquad |-\rangle_4 = \cos\theta|0\rangle_4 - \sin\theta|1\rangle_4, \tag{2}$$

(where θ is a measured angle with the region $0 < \theta \leq \pi/4$) and informs his measurement result to Alice, David and Bob through a classical channel. It is noted that the non-maximally entangled state in the new basis $\{|+\rangle_4, |-\rangle_4\}$ can be rewritten as

$$|\xi\rangle_{1234} = |\psi\rangle_{123} \otimes |+\rangle_4 + |\varphi\rangle_{123} \otimes |-\rangle_4, \tag{3}$$

where

$$\begin{aligned} |\psi\rangle_{123} &= (\alpha\sin\theta + \beta\cos\theta)|000\rangle_{123} + (\gamma\sin\theta + \delta\cos\theta)|111\rangle_{123}, \\ |\varphi\rangle_{123} &= (\alpha\cos\theta - \beta\sin\theta)|000\rangle_{123} + (\gamma\cos\theta - \delta\sin\theta)|111\rangle_{123} \end{aligned} \tag{4}$$

are unnormalized state vectors and their norms stand for the absolute probabilities for each case. Obviously, corresponding to Cliff's measurement result $|+\rangle_4$ or $|-\rangle_4$, the state of particles 1, 2 and 3 collapses to $|\psi\rangle_{123}$ or $|\varphi\rangle_{123}$, respectively. Generally, the state $|\psi\rangle_{123}$ and $|\varphi\rangle_{123}$ are not maximally entangled, and the success probability of dense coding with them is less than 1.

Now let us first analyze the case where Cliff's measurement gives $|+\rangle_4$ and the state of particles 1, 2 and 3 collapses to $|\psi\rangle_{123}$. After receiving Cliff's measurement result, Alice first introduces an auxiliary particle with initial state $|0\rangle_{aux}$ and performs a unitary transformation

$$U_1 = \begin{pmatrix} \frac{\gamma\sin\theta+\delta\cos\theta}{\alpha\sin\theta+\beta\cos\theta} & 0 & \sqrt{1-\frac{(\gamma\sin\theta+\delta\cos\theta)^2}{(\alpha\sin\theta+\beta\cos\theta)^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\frac{(\gamma\sin\theta+\delta\cos\theta)^2}{(\alpha\sin\theta+\beta\cos\theta)^2}} & 0 & -\frac{\gamma\sin\theta+\delta\cos\theta}{\alpha\sin\theta+\beta\cos\theta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

on the auxiliary particle and particle 1 under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$. The collective unitary transformation $U_1 \otimes I_1$ transforms the state $|0\rangle_{aux} \otimes |\psi\rangle_{123}$ to

$$\begin{aligned} |\psi\rangle_{aux23} &= \sqrt{2}(\gamma\sin\theta+\delta\cos\theta)|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|000\rangle_{123}+|111\rangle_{123})\right] \\ &\quad + \sqrt{(\alpha\sin\theta+\beta\cos\theta)^2-(\gamma\sin\theta+\delta\cos\theta)^2}|1\rangle_{aux} \otimes |000\rangle_{123}. \end{aligned} \tag{6}$$

Then Alice performs a von Neumann measurement on the auxiliary particle under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If she obtains $|0\rangle_{aux}$, particles 1, 2 and 3 are maximally entangled. Alice now performs one of the four unitary transformations $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on particle 1 and sends it to Bob. However, if so does David, Bob cannot completely discriminate Alice's and David's operation, it is necessary to restrict the operations of Alice's or David's. In this letter, we assume that Alice can make any one of the four unitary operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her particle, while David makes any one of $\{I, \sigma_X\}$ on his particle. After Alice's and David's operations, Bob can discriminate Alice's and David's unitary transformations on their particles by performing a joint measurement on particles 1, 2 and 3, and 3 bits of information are transmitted. However, if Alice obtains $|1\rangle_{aux}$, particles 1, 2 and 3 are unentangled. Bob can extract only 2 bit of information. So, in this case, on average

$$\begin{aligned} I_1 &= 2(\gamma\sin\theta+\delta\cos\theta)^2 \times 3 \\ &\quad + \left[(\alpha\sin\theta+\beta\cos\theta)^2-(\gamma\sin\theta+\delta\cos\theta)^2\right] \times 2 \\ &= 4(\gamma\sin\theta+\delta\cos\theta)^2 + 2(\alpha\sin\theta+\beta\cos\theta)^2 \end{aligned} \tag{7}$$

bits of information are transmitted from Alice to Bob.

If Cliff's measurement result is $|-\rangle_4$, Alice's unitary transformation on the auxiliary particle and particle 1 should be divided into two cases:

1. If the angle θ satisfies $\alpha\cos\theta-\beta\sin\theta \geq \gamma\cos\theta-\delta\sin\theta$, then Alice's unitary transformation is

$$U_2 = \begin{pmatrix} \frac{\gamma\cos\theta-\delta\sin\theta}{\alpha\cos\theta-\beta\sin\theta} & 0 & \sqrt{1-\frac{(\gamma\cos\theta+\delta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\frac{(\gamma\cos\theta-\delta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2}} & 0 & -\frac{\gamma\cos\theta-\delta\sin\theta}{\alpha\cos\theta-\beta\sin\theta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

The collective unitary transformation $U_2 \otimes I_1$ transforms the state $|0\rangle_{aux} \otimes |\varphi\rangle_{123}$ to

$$\begin{aligned} |\psi\rangle_{aux23} &= \sqrt{2}(\gamma\cos\theta-\delta\sin\theta)|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|000\rangle_{123}+|111\rangle_{123})\right] \\ &\quad + \sqrt{(\alpha\cos\theta-\beta\sin\theta)^2-(\gamma\cos\theta+\delta\sin\theta)^2}|1\rangle_{aux} \otimes |000\rangle_{123}. \end{aligned} \tag{9}$$

Then Alice performs a von Neumann measurement on the auxiliary particle under the basis $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If Alice gets the result $|1\rangle_{aux}$, the state of particles 1, 2, 3 is unentangled, and 2 bit information can be transmitted. If she gets $|0\rangle_{aux}$, the state of particles 1, 2 and 3 is maximally entangled, and 3 bits information can be transmitted. So in the case, Alice and David can transmit

$$\begin{aligned} I_2 &= 2(\gamma\cos\theta - \delta\sin\theta)^2 \times 3 \\ &\quad + \left[(\alpha\cos\theta - \beta\sin\theta)^2 - (\gamma\cos\theta - \delta\sin\theta)^2\right] \times 2 \\ &= 4(\gamma\cos\theta - \delta\sin\theta)^2 + 2(\alpha\cos\theta - \beta\sin\theta)^2 \end{aligned} \tag{10}$$

bits of information on average.

2. If $\gamma\cos\theta - \delta\sin\theta \geq \alpha\cos\theta - \beta\sin\theta$, then the corresponding unitary transformation is

$$U_2' = \begin{pmatrix} \frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta} & 0 & \sqrt{1-\frac{(\alpha\cos\theta-\beta\sin\theta)^2}{(\gamma\cos\theta+\delta\sin\theta)^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\frac{(\alpha\cos\theta-\beta\sin\theta)^2}{(\gamma\cos\theta-\delta\sin\theta)^2}} & 0 & -\frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

The collective unitary transformation $U_2 \otimes I_1$ transforms the state $|0\rangle_{aux} \otimes |\varphi\rangle_{123}$ to

$$\begin{aligned} |\psi\rangle_{aux23} &= \sqrt{2}(\alpha\cos\theta - \beta\sin\theta)|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|000\rangle_{123} + |111\rangle_{123})\right] \\ &\quad + \sqrt{(\gamma\cos\theta + \delta\sin\theta)^2 - (\alpha\cos\theta - \beta\sin\theta)^2}|1\rangle_{aux} \otimes |000\rangle_{123}, \end{aligned} \tag{12}$$

and the senders can transmit

$$\begin{aligned} I_2' &= 2(\alpha\cos\theta - \beta\sin\theta)^2 \times 3 \\ &\quad + \left[(\gamma\cos\theta - \delta\sin\theta)^2 - (\alpha\cos\theta - \beta\sin\theta)^2\right] \times 2 \\ &= 4(\alpha\cos\theta - \beta\sin\theta)^2 + 2(\gamma\cos\theta - \delta\sin\theta)^2 \end{aligned} \tag{13}$$

bits of information to Bob.

Synthesizing all the measurements, the average amount of information transmitted from Alice and David to Bob adds up to

$$I = \begin{cases} I_1 + I_2 = 2 + 2(\gamma^2 + \delta^2), & \frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta} \geq 1, \\ I_1 + I_2' = 2 + 2[(\gamma\sin\theta + \delta\cos\theta)^2 + (\alpha\cos\theta - \beta\sin\theta)^2], & \frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta} \leq 1. \end{cases} \tag{14}$$

bits.

From (14) we can see that the average amount of information transmitted from Alice and David to Bob only depends on Cliff's measurement angle θ. Therefore, it is helpful for Cliff to control the average amount of information transmitted from Alice and David to Bob by only adjusting the measurement angle θ.

3 Dense Coding with Generalized Measurement

In this section, we first consider the case in which Cliff's measurement is $|+\rangle_4$, and the other case can be deduced in a similar fashion lately. After receiving the measurement result, Alice doesn't concentrate the channel, she directly uses any one of the four unitary operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her particle 1 and David makes any one of $\{I, \sigma_X\}$ on his particle 2. After the senders' unitary transformations, the shared state $|\psi\rangle_{123}$ undergoes one of the following transformations:

$$\begin{aligned}
(I\otimes I\otimes I)|\psi\rangle_{123} &= (\alpha\sin\theta+\beta\cos\theta)|000\rangle_{123}+(\gamma\sin\theta+\delta\cos\theta)|111\rangle_{123}=|\sigma\rangle_1,\\
(\sigma_X\otimes I\otimes I)|\psi\rangle_{123} &= (\alpha\sin\theta+\beta\cos\theta)|100\rangle_{123}+(\gamma\sin\theta+\delta\cos\theta)|011\rangle_{123}=|\sigma\rangle_2,\\
(i\sigma_Y\otimes I\otimes I)|\psi\rangle_{123} &= -(\alpha\sin\theta+\beta\cos\theta)|100\rangle_{123}+(\gamma\sin\theta+\delta\cos\theta)|011\rangle_{123}=|\sigma\rangle_3,\\
(\sigma_Z\otimes I\otimes I)|\psi\rangle_{123} &= (\alpha\sin\theta+\beta\cos\theta)|000\rangle_{123}-(\gamma\sin\theta+\delta\cos\theta)|111\rangle_{123}=|\sigma\rangle_4,\\
(I\otimes \sigma_X\otimes I)|\psi\rangle_{123} &= (\alpha\sin\theta+\beta\cos\theta)|010\rangle_{123}+(\gamma\sin\theta+\delta\cos\theta)|101\rangle_{123}=|\sigma\rangle_5,\\
(\sigma_X\otimes \sigma_X\otimes I)|\psi\rangle_{123} &= (\alpha\sin\theta+\beta\cos\theta)|110\rangle_{123}+(\gamma\sin\theta+\delta\cos\theta)|001\rangle_{123}=|\sigma\rangle_6,\\
(i\sigma_Y\otimes \sigma_X\otimes I)|\psi\rangle_{123} &= -(\alpha\sin\theta+\beta\cos\theta)|110\rangle_{123}+(\gamma\sin\theta+\delta\cos\theta)|001\rangle_{123}=|\sigma\rangle_7,\\
(\sigma_Z\otimes \sigma_X\otimes I)|\psi\rangle_{123} &= (\alpha\sin\theta+\beta\cos\theta)|010\rangle_{123}-(\gamma\sin\theta+\delta\cos\theta)|101\rangle_{123}=|\sigma\rangle_8.
\end{aligned} \tag{15}$$

Then Alice and David send their particles to Bob, and now Bob has at his disposal three particles which could be in any one of the eight possible states $\{|\sigma\rangle_i, i=1,2,\ldots,8\}$. If Bob is able to distinguish all the eight nonorthogonal states conclusively, he can extract 3 bits of information. However, the above 8 states are not mutually orthogonal, so these 8 non-orthogonal states cannot be distinguished with certainty. But they can be distinguished with some probability [25–28].

In order to distinguish the above set $\{|\sigma\rangle_i, i=1,2,\ldots,8\}$, Bob first performs a projection onto the subspaces spanned by the bases states $\{|000\rangle,|111\rangle\}$, $\{|001\rangle,|110\rangle\}$, $\{|010\rangle,|101\rangle\}$ and $\{|100\rangle,|011\rangle\}$, with corresponding projective operators are $P_1=|000\rangle\langle 000|+|111\rangle\langle 111|$, $P_2=|001\rangle\langle 001|+|110\rangle\langle 110|$, $P_3=|010\rangle\langle 010|+|101\rangle\langle 101|$ and $P_4=|100\rangle\langle 100|+|011\rangle\langle 011|$, respectively. Obviously, the above four subspaces are mutually orthogonal, so Bob can discriminate the four subsets of Alice's and David's operators: $\{I\otimes I, \sigma_Z\otimes I\}$, $\{\sigma_X\otimes\sigma_X, i\sigma_Y\otimes\sigma_X\}$, $\{I\otimes\sigma_X, \sigma_Z\otimes\sigma_X\}$ and $\{\sigma_X\otimes I, i\sigma_Y\otimes I\}$, that is to say, he gets 2 bit of information [25, 29]. If Bob obtains P_1, then he knows that the state will be either $|\sigma\rangle_1$ or $|\sigma\rangle_4$. Similarly, if he obtains P_2, the state will be either $|\sigma\rangle_6$ or $|\sigma\rangle_7\ldots$. Suppose Bob obtains P_1, then he performs a generalized measurement on his three-particle states with the corresponding positive operator valued measure (POVM) elements are

$$\begin{aligned}
M_1 &= \frac{1}{2}\begin{bmatrix} \frac{(\gamma\sin\theta+\delta\cos\theta)^2}{(\alpha\sin\theta+\beta\cos\theta)^2} & \frac{\gamma\sin\theta+\delta\cos\theta}{\alpha\sin\theta+\beta\cos\theta} \\ \frac{\gamma\sin\theta+\delta\cos\theta}{\alpha\sin\theta+\beta\cos\theta} & 1 \end{bmatrix},\\
M_2 &= \frac{1}{2}\begin{bmatrix} \frac{(\gamma\sin\theta+\delta\cos\theta)^2}{(\alpha\sin\theta+\beta\cos\theta)^2} & -\frac{\gamma\sin\theta+\delta\cos\theta}{\alpha\sin\theta+\beta\cos\theta} \\ -\frac{\gamma\sin\theta+\delta\cos\theta}{\alpha\sin\theta+\beta\cos\theta} & 1 \end{bmatrix},\\
M_3 &= \begin{bmatrix} 1-\frac{(\gamma\sin\theta+\delta\cos\theta)^2}{(\alpha\sin\theta+\beta\cos\theta)^2} & 0 \\ 0 & 0 \end{bmatrix}\sigma.
\end{aligned} \tag{16}$$

If Bob gets M_1 then the state is $|\sigma_1\rangle$, if he gets M_2 then the state is $|\sigma_4\rangle$. However if he gets M_3 the state is completely indecisive and Bob cannot obtain any information. The success probability of distinguishing $|\sigma_1\rangle$ and $|\sigma_4\rangle$ is $\frac{2(\gamma\sin\theta+\delta\cos\theta)^2}{(\gamma\sin\theta+\delta\cos\theta)^2+(\alpha\sin\theta+\beta\cos\theta)^2}$, which is also the probability that Bob obtains another 1 bit of information. Similar procedure can be applied for the case of M_2, one can easily check that the relevant POVM elements and the success probability are the same. So, in this case, the average amount of information transmitted from Alice and David to Bob should be expressed as

$$\begin{aligned} I_{1'} &= \left[(\gamma\sin\theta+\delta\cos\theta)^2+(\alpha\sin\theta+\beta\cos\theta)^2\right] \\ &\quad\times\left[2+\frac{2(\gamma\sin\theta+\delta\cos\theta)^2}{(\gamma\sin\theta+\delta\cos\theta)^2+(\alpha\sin\theta+\beta\cos\theta)^2}\times 1\right] \\ &= 4(\gamma\sin\theta+\delta\cos\theta)^2+2(\alpha\sin\theta+\beta\cos\theta)^2. \end{aligned} \tag{17}$$

If Charlie's measurement result is $|-\rangle_4$, then the state of particles 1, 2 and 3 collapses to $|\varphi\rangle_{123}$. After Alice's encoding with one of the four operators $\{I,\sigma_X,i\sigma_Y,\sigma_Z\}$, and David's encoding with one of the two operators $\{I,\sigma_X\}$, the state $|\varphi\rangle_{123}$ becomes one of the following forms:

$$\begin{aligned} (I\otimes I\otimes I)|\varphi\rangle_{123} &= (\alpha\cos\theta-\beta\sin\theta)|000\rangle_{123}+(\gamma\cos\theta-\delta\sin\theta)|111\rangle_{123}=|\varrho\rangle_1, \\ (\sigma_X\otimes I\otimes I)|\varphi\rangle_{123} &= (\alpha\cos\theta-\beta\sin\theta)|100\rangle_{123}+(\gamma\cos\theta-\delta\sin\theta)|011\rangle_{123}=|\varrho\rangle_2, \\ (i\sigma_Y\otimes I\otimes I)|\varphi\rangle_{123} &= -(\alpha\cos\theta-\beta\sin\theta)|100\rangle_{123}+(\gamma\cos\theta-\delta\sin\theta)|011\rangle_{123}=|\varrho\rangle_3, \\ (\sigma_Z\otimes I\otimes I)|\varphi\rangle_{123} &= (\alpha\cos\theta-\beta\sin\theta)|000\rangle_{123}-(\gamma\cos\theta-\delta\sin\theta)|111\rangle_{123}=|\varrho\rangle_4, \\ (I\otimes \sigma_X\otimes I)|\varphi\rangle_{123} &= (\alpha\cos\theta-\beta\sin\theta)|010\rangle_{123}+(\gamma\cos\theta-\delta\sin\theta)|101\rangle_{123}=|\varrho\rangle_5, \\ (\sigma_X\otimes \sigma_X\otimes I)|\varphi\rangle_{123} &= (\alpha\cos\theta-\beta\sin\theta)|110\rangle_{123}+(\gamma\cos\theta-\delta\sin\theta)|001\rangle_{123}=|\varrho\rangle_6, \\ (i\sigma_Y\otimes \sigma_X\otimes I)|\varphi\rangle_{123} &= -(\alpha\cos\theta-\beta\sin\theta)|110\rangle_{123}+(\gamma\cos\theta-\delta\sin\theta)|001\rangle_{123}=|\varrho\rangle_7, \\ (\sigma_Z\otimes \sigma_X\otimes I)|\varphi\rangle_{123} &= (\alpha\cos\theta-\beta\sin\theta)|010\rangle_{123}-(\gamma\cos\theta-\delta\sin\theta)|101\rangle_{123}=|\varrho\rangle_8. \end{aligned} \tag{18}$$

Bob's projection measurement is the same as before, but the corresponding POVM elements should be treated with two cases:

1. If Cliff's measurement angle θ satisfies $\alpha\cos\theta-\beta\sin\theta\geq\gamma\cos\theta-\delta\sin\theta$, then the POVM elements in the subspace $\{|000\rangle,|111\rangle\}$ are

$$\begin{aligned} M_1' &= \frac{1}{2}\begin{bmatrix} \frac{(\gamma\cos\theta-\delta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2} & \frac{\gamma\cos\theta-\delta\sin\theta}{\alpha\cos\theta-\beta\sin\theta} \\ \frac{\gamma\cos\theta-\delta\sin\theta}{\alpha\cos\theta-\beta\sin\theta} & 1 \end{bmatrix}, \\ M_2' &= \frac{1}{2}\begin{bmatrix} \frac{(\gamma\cos\theta-\delta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2} & \frac{\delta\sin\theta-\gamma\cos\theta}{\alpha\cos\theta-\beta\sin\theta} \\ \frac{\delta\sin\theta-\gamma\cos\theta}{\alpha\cos\theta-\beta\sin\theta} & 1 \end{bmatrix}, \\ M_3' &= \begin{bmatrix} 1-\frac{(\gamma\cos\theta-\delta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{19}$$

In this case, Bob can discriminate $|\varrho_1\rangle$ from $|\varrho_4\rangle$ with probability $\frac{2(\gamma\cos\theta-\delta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2+(\gamma\cos\theta-\delta\sin\theta)^2}$.

That is to say, Bob can obtain

$$\begin{aligned} I_{2'} &= \left[(\alpha\cos\theta-\beta\sin\theta)^2+(\gamma\cos\theta-\delta\sin\theta)^2\right] \\ &\quad\times\left[2+\frac{2(\gamma\cos\theta-\delta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2+(\gamma\cos\theta-\delta\sin\theta)^2}\times 1\right] \\ &= 4(\gamma\cos\theta-\delta\sin\theta)^2+2(\alpha\cos\theta-\beta\sin\theta)^2 \end{aligned} \tag{20}$$

bits of information from Alice and David.

2. If $\gamma\cos\theta-\delta\sin\theta\geq\alpha\cos\theta-\beta\sin\theta$, then the POVM elements in the subspace $\{|000\rangle, |111\rangle\}$ are

$$\begin{aligned} M_1'' &= \frac{1}{2}\begin{bmatrix} \frac{(\alpha\cos\theta-\beta\sin\theta)^2}{(\gamma\cos\theta-\delta\sin\theta)^2} & \frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta} \\ \frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta} & 1 \end{bmatrix}, \\ M_2'' &= \frac{1}{2}\begin{bmatrix} \frac{(\alpha\cos\theta-\beta\sin\theta)^2}{(\gamma\cos\theta-\delta\sin\theta)^2} & \frac{\beta\sin\theta-\alpha\cos\theta}{\gamma\cos\theta-\delta\sin\theta} \\ \frac{\beta\sin\theta-\alpha\cos\theta}{\gamma\cos\theta-\delta\sin\theta} & 1 \end{bmatrix}, \\ M_3'' &= \begin{bmatrix} 1-\frac{(\alpha\cos\theta-\beta\sin\theta)^2}{(\gamma\cos\theta-\delta\sin\theta)^2} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{21}$$

In this case, Bob can discriminate $|\varrho_1\rangle$ from $|\varrho_4\rangle$ with probability $\frac{2(\alpha\cos\theta-\beta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2+(\gamma\cos\theta-\delta\sin\theta)}$. So Bob can extract

$$\begin{aligned} I_{2''} &= \left[(\alpha\cos\theta-\beta\sin\theta)^2+(\gamma\cos\theta-\delta\sin\theta)^2\right] \\ &\quad\times\left[2+\frac{2(\alpha\cos\theta-\beta\sin\theta)^2}{(\alpha\cos\theta-\beta\sin\theta)^2+(\gamma\cos\theta-\delta\sin\theta)^2}\times 1\right] \\ &= 4(\alpha\cos\theta-\beta\sin\theta)^2+2(\gamma\cos\theta-\delta\sin\theta)^2 \end{aligned} \tag{22}$$

bits of information from Alice and David.

It is easy to check that Bob can discriminate the other states with the same procedure and the same probability. Synthesizing all the cases, the average amount of information transmitted from the senders to Bob can be written as

$$I=\begin{cases} I_{1'}+I_{2'}=2+2(\gamma^2+\delta^2), & \frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta}\geq 1, \\ I_{1'}+I_{2''}=2+2[(\gamma\sin\theta+\delta\cos\theta)^2+(\alpha\cos\theta-\beta\sin\theta)^2], & \frac{\alpha\cos\theta-\beta\sin\theta}{\gamma\cos\theta-\delta\sin\theta}\leq 1. \end{cases} \tag{23}$$

Comparing (14) with (23), we can see that the two schemes have the same transmission capacity, which means that the two schemes are equivalent for the controlled dense coding and the results are unique. From the above discussion, we can also see that the second scheme is advantageous over the first one because it needn't introduce extra auxiliary particle, i.e. it make less use of the physical resource than the first method.

4 Summary

In summary, two schemes, with entanglement concentration and generalized measurement respectively, of realizing controlled dense coding are investigated with a four-particle non-maximally entangled state. It is shown that Cliff can control the average amount of information by only adjusting the measured angle θ. It is also shown that the results for the average amounts of information are unique from the two different schemes.

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